**First part: multiple choice questions**

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $(a_n)_{n \geq 1}$ be the sequence defined by

$$a_n = (-1)^n \left(\frac{6n+8}{2n} \right) - 3 - \frac{4}{n}.$$

Then:

- ☒ $\liminf_{n \rightarrow +\infty} a_n = -6$, and $\limsup_{n \rightarrow +\infty} a_n = 0$
- ☐ $\liminf_{n \rightarrow +\infty} a_n = -14$, and $\limsup_{n \rightarrow +\infty} a_n = 0$
- ☐ $\liminf_{n \rightarrow +\infty} a_n = -6$, and $\limsup_{n \rightarrow +\infty} a_n = 6$
- ☐ $\liminf_{n \rightarrow +\infty} a_n = -3$, and $\limsup_{n \rightarrow +\infty} a_n = 0$

Question 2 : Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded subsets. Then

- ☒ $\sup(A \cup B) = \sup A \cdot \sup B$ ☒ $\sup(A \cup B) = \max\{\sup A, \sup B\}$
- ☐ $\sup(A \cup B) = \min\{\sup A, \sup B\}$ ☐ $\sup(A \cup B) = \sup A + \sup B$

Question 3 : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |x| & \text{if } x \geq -1, \\ \frac{1}{2}(x^2 + 1) & \text{if } x < -1. \end{cases}$$

Then:

- ☐ f is differentiable on \mathbb{R}
- ☐ f is not continuous at $x = -1$
- ☐ f is differentiable at 0 and continuous at $x = -1$
- ☒ f is differentiable at $x = -1$ and continuous at $x = 0$

Question 4 : Let $I = [0, \frac{\pi}{2}]$ and $f: I \rightarrow \mathbb{R}$ be the function defined by $f(x) = \cos(2x)$. Then for each $x, y \in I$ such that $x < y$ we have:

- ☒ $-2 \leq \frac{f(y) - f(x)}{y - x} \leq 0$ ☐ $-1 \leq \frac{f(y) - f(x)}{y - x} \leq 1$
- ☐ $-\pi \leq \frac{f(y) - f(x)}{y - x} \leq -1$ ☐ $0 \leq \frac{f(y) - f(x)}{y - x} \leq 2$

Question 5 : The integral $\int_0^1 \frac{2x-1}{(x-3)(x+2)} dx$ equals:

- ☒ 0 ☐ $\sqrt{6} \arctan\left(\frac{1}{6}\right)$
- ☐ -1 ☐ $\log(3) - \log(2)$



Question 6 : Let $f: [\frac{1}{2}, 1] \rightarrow \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x} + \frac{1}{\pi} \sin(\frac{\pi}{x})$. Let I be the range of f . Then:

☐ $I = [1, 1 + \frac{1}{\pi}]$

☐ $I = [1 - \frac{1}{\pi}, 1]$

☐ $I = [2, 3]$

☒ $I = [1, 2]$

Question 7 : Let $\alpha \in \mathbb{R}$. The real series $\sum_{n=1}^{\infty} \left(1 + \frac{\alpha}{n}\right)^{n^2}$ converges if and only if

☐ $-1 < \alpha < 0$

☐ $\alpha < -1$

☐ $\alpha \geq 0$

☒ $\alpha < 0$

Question 8 : The generalized integral $I = \int_{0+}^1 \frac{\log x}{x^2} dx$

☐ converges and equals -1

☐ converges and equals 1

☐ converges and equals -4

☒ diverges

Question 9 :

The complex numbers $3, 1 - 2i$, and $1 + 2i$ are the roots (the zeros) of the polynomial

☒ $z^3 - 5z^2 + 11z - 15$

☐ $z^3 - 2iz^2 + 45$

☐ $z^3 + 14z^2 + 15$

☐ $z^3 - 5z^2 + 5z + 45$

Question 10 : Let $a_0 \in \mathbb{R}$ and $(a_n)_{n \geq 0}$ a sequence of real numbers satisfying the following recurrence relation for $n \geq 1$

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{2}.$$

Then:

☐ if $a_0 < 0$, $\lim_{n \rightarrow \infty} a_n = -\infty$

☒ if $a_0 = 0$ the sequence is convergent

☐ if $a_0 > 1$ the sequence is increasing

☐ if $a_0 < 1$ the sequence is decreasing

Question 11 : Consider the series $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = 1$ if n is even and $a_n = 0$ if n is odd.

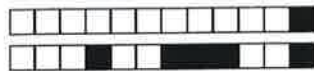
The radius of convergence for the series, R , satisfies

☐ $R = \frac{1}{2}$

☒ $R = 1$

☐ $R = \infty$

☐ $R = 0$



Question 12 : Let $(a_n)_{n \geq 1}$ be the sequence defined by

$$a_n = (3n + 1)^{\log(\frac{1}{\sqrt{n}})}.$$

Then:

☐ $\lim_{n \rightarrow +\infty} a_n = 1$

☐ $\lim_{n \rightarrow +\infty} a_n = +\infty$

☒ $\lim_{n \rightarrow +\infty} a_n = 0$

☐ $\lim_{n \rightarrow +\infty} a_n = 3$

Question 13 : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-2/|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

☐ $\lim_{x \rightarrow 0} f(x)$ exists but f is not continuous at 0

☐ $\lim_{x \rightarrow 0} f(x)$ does not exist

☐ f is continuous at $x = 0$ but not differentiable at $x = 0$

☒ f is differentiable at $x = 0$

Question 14 : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = e^x \log(1+x)$. The expansion of order 3 of f at $x_0 = 0$ is

☐ $f(x) = x + \frac{x^2}{2} + \frac{x^3}{2} + x^3 \varepsilon_3(x)$

☐ $f(x) = x - \frac{x^2}{3} + \frac{x^3}{2} + x^3 \varepsilon_3(x)$

☒ $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + x^3 \varepsilon_3(x)$

☐ $f(x) = x + \frac{x^2}{3} - \frac{x^3}{2} + x^3 \varepsilon_3(x)$

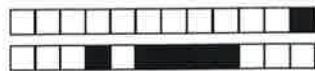
Question 15 : Let the sequence $\{a_n\}_{n \geq 0}$ be defined by recurrence with $a_0 = \frac{3}{2}$ and for $n \geq 1$, $a_n = 3 - \frac{2}{a_{n-1}}$, then:

☐ the limit does not exist in \mathbb{R}

☒ $\lim_{n \rightarrow \infty} a_n = 2$

☐ $\lim_{n \rightarrow \infty} a_n = 1$

☐ $\lim_{n \rightarrow \infty} a_n = 4$



Question 16 : Let $a, b \in \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{\sqrt{2}}{2} & \text{si } x \leq 0, \\ \sin(ax + b) & \text{si } x > 0. \end{cases}$$

Then, f is continuous on \mathbb{R} for :

☐ $a = \frac{\pi}{2}$ et $b = \frac{\pi}{2}$

☐ $a = 0$ et $b = -\frac{\pi}{4}$

☐ $a = -\frac{\pi}{4}$ et $b = 0$

☒ $a = 0$ et $b = \frac{\pi}{4}$

Question 17 : For all $k \in \mathbb{N}$, $k \geq 1$, let $a_k = (-1)^k \frac{k+1}{k^2}$ and $s_n = \sum_{k=1}^n a_k$. Then

☒ the series $\sum_{k=1}^{\infty} a_k$ converges, but not absolutely.

☐ the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

☐ $\lim_{n \rightarrow \infty} s_n = \infty$.

☐ $\lim_{n \rightarrow \infty} s_n = -\infty$.

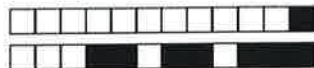
Question 18 : The integral $\int_0^1 x^2 e^{-x} dx$ equals

☒ $2 - \frac{5}{e}$

☐ $2 - \frac{1}{e}$

☐ $2 - \frac{3}{e}$

☐ $2 - \frac{4}{e}$

**Second part: true/false questions**

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $(a_n)_{n \geq 0}$ be a sequence of non zero real numbers such that $\lim_{n \rightarrow \infty} a_n = 2$. Then $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$.

☒ TRUE ☐ FALSE

Question 20 : Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a bijective increasing function. Then $f^{-1}: \mathbb{R} \rightarrow \mathbb{R}$ is increasing.

☒ TRUE ☐ FALSE

Question 21 : The definite integral $\int_{-1}^1 e^{-\sin(x)} dx$ is equal to zero.

☐ TRUE ☒ FALSE

Question 22 : Let $f \in C^\infty(\mathbb{R})$. Then for each $x_0 \in \mathbb{R}$ and each $n \in \mathbb{N}^*$ there is a Taylor expansion for f of order n around x_0 .

☒ TRUE ☐ FALSE

Question 23 : Let $f \in C^1(\mathbb{R})$. Then there exist numbers $a, b \in \mathbb{R}$ such that

$$\lim_{x \rightarrow 0} \frac{f(x) - a - bx}{x} = 0$$

☒ TRUE ☐ FALSE

Question 24 : For $z \in \mathbb{C}^*$, $z^5 + \frac{1}{z^5}$ is real if $|z| = 1$.

☒ TRUE ☐ FALSE

Question 25 : Let $f: [0, 1] \rightarrow [0, 1]$ be a continuous function with range $[0, 1]$. Then, there must exist $x \in [0, 1]$ such that $f(x) - x = 0$.

☒ TRUE ☐ FALSE



Question 26 : Let $A, B \subseteq \mathbb{R}$ be two non empty bounded subsets. If $\inf A \leq \inf B$ and $\sup A \geq \sup B$, then $B \subseteq A$.

☐

TRUE

☒

FALSE

Question 27 : Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function that is continuous at $x_0 = 0$. Then, the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = xf(x)$ is differentiable at $x_0 = 0$.

☒

TRUE

☐

FALSE

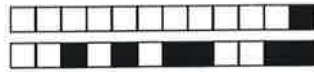
Question 28 : Let $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}$ be two sequences of real numbers such that the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge. Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

☐

TRUE

☒

FALSE



Question 30: This question is worth 5 points.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

Do not write here.

Show by induction that for each $n \geq 1$, $\sum_{k=1}^n 3^k(k + \frac{1}{2}) = 3^n \frac{3n}{2}$.

Let $H(n)$ be $\sum_{k=1}^n 3^k(k + \frac{1}{2}) = 3^n \frac{3n}{2}$.

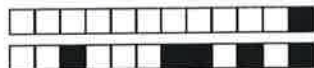
We must show a) $H(1)$ is true b) $H(n) \Rightarrow H(n+1)$.

a) (INITIAL STEP) IF $n=1$. LHS = $3^1(1 + \frac{1}{2}) = 3 \cdot \frac{3}{2}$,
 $= 3^1 \frac{3 \times 1}{2} = \text{RHS}$.

b) (INDUCTIVE STEP): SUPPOSE $H(n)$ IS TRUE.

Then $\sum_{k=1}^{n+1} 3^k(k + \frac{1}{2}) = \sum_{k=1}^n 3^k(k + \frac{1}{2}) + 3^{n+1}(n+1 + \frac{1}{2})$
 $= \sum_{k=1}^n 3^k(k + \frac{1}{2}) + 3^{n+1}(n + \frac{3}{2})$
 $\stackrel{H(n)}{=} 3^n \left(\frac{3n}{2}\right) + 3^{n+1}(n + \frac{3}{2}) = 3^{n+1} \left(\frac{n}{2} + n + \frac{3}{2}\right)$
 $= 3^{n+1} \left(\frac{3n}{2} + \frac{3}{2}\right) = 3^{n+1} \frac{3(n+1)}{2}$

i.e. $H(n+1)$ IS TRUE

**Third part, open questions**

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: This question is worth 6 points.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 ☐ 6

Do not write here.

- (a) State the Bolzano Weierstrass theorem.
- (b) Give a result for continuous functions on closed bounded intervals whose proof relies on the Bolzano Weierstrass Theorem. (No proof required.)
- (c) Give an example of two sequences $\{a_n\}_{n>0}$, $\{b_n\}_{n>0}$ for which $\limsup_{n \rightarrow \infty} (a_n + b_n) < \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$.

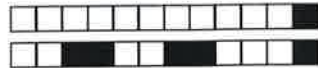
a) IF $(a_n)_{n>1}$ IS A BOUNDED SEQUENCE THEN
 \exists A CONVERGENT SUBSEQUENCE OF $(a_n)_{n>1}$

b) IF f IS A CTS FUNCTION ON A CLOSED BOUNDED INTERVAL I THEN $\sup_{x \in I} f(x) < \infty$.
(OR $\exists x_0 \in I$ SUCH THAT $f(x_0) = \sup_{x \in I} f(x)$
OR f IS UNIFORMLY CTS).

c). LET $a_n = (-1)^n$ $b_n = (-1)^{n+1}$ $n > 0$.

THEN $a_n + b_n = 0 \forall n$ SO

$$\limsup (a_n + b_n) = 0 < \limsup a_n + \limsup b_n = 1 + 1$$



Question 31: This question is worth 5 points.

☐ 0 ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

Do not write here.

Show that the equation $\cos(x) + x = 5$ has a unique solution in \mathbb{R} .

$f(x) = \cos(x) + x$ has $f'(x) = -\sin(x) + 1 \geq 0$.

so $f'(x) = 0$ only at $\pi/2 + 2n\pi$. otherwise it is strictly positive.

so by MVT for diff functions.

f is strictly increasing on intervals

$$I_n = \left[\pi/2 + 2n\pi, \pi/2 + 2(n+1)\pi \right]$$

so f is strictly increasing on $\cup I_n = \mathbb{R}$.

ie f is strictly increasing so \exists at most 1 value in \mathbb{R} with $f(x) = 5$. (*)

but $f(0) = 1$ $f(7) \geq 6$ so by the

intermediate value theorem for CB functions

$\exists x \in]0, 7[$ with $f(x) = 5$.

by (*) this x is unique in \mathbb{R} . \square

