First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $(a_n)_{n\geq 1}$ be the sequence defined by

$$a_n = (-1)^n \left(\frac{6n+8}{2n}\right) - 3 - \frac{4}{n}.$$

Then:

- [x] $\liminf\limits_{n\to+\infty} a_n = -6$, and $\limsup\limits_{n\to+\infty} a_n = 0$
- [ ] $\liminf\limits_{n\to+\infty} a_n = -14$, and $\limsup\limits_{n\to+\infty} a_n = 0$
- [ ] $\liminf\limits_{n\to+\infty} a_n = -6$, and $\limsup\limits_{n\to+\infty} a_n = 6$
- [ ] $\liminf\limits_{n\to+\infty} a_n = -3$, and $\limsup\limits_{n\to+\infty} a_n = 0$

Question 2 : Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded subsets. Then

- [ ] $\sup(A \cup B) = \sup A \cdot \sup B$
- [x] $\sup(A \cup B) = \max\{\sup A, \sup B\}$
- [ ] $\sup(A \cup B) = \min\{\sup A, \sup B\}$
- [ ] $\sup(A \cup B) = \sup A + \sup B$

Question 3 : Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |x| & \text{if } x \geq -1\,, \\ \frac{1}{2}(x^2+1) & \text{if } x < -1\,. \end{cases}$$

Then:

- [ ] f is differentiable on $\mathbb{R}$
- [ ] f is not continuous at $x = -1$
- [ ] f is differentiable at 0 and continuous at $x = -1$
- [x] f is differentiable at $x = -1$ and continuous at $x = 0$

Question 4 : Let $I = \left[0, \frac{\pi}{2}\right]$ and $f : I \to \mathbb{R}$ be the function defined by $f(x) = \cos(2x)$. Then for each $x, y \in I$ such that $x < y$ we have:

- [x] $-2 \leq \dfrac{f(y)-f(x)}{y-x} \leq 0$
- [ ] $-1 \leq \dfrac{f(y)-f(x)}{y-x} \leq 1$
- [ ] $-\pi \leq \dfrac{f(y)-f(x)}{y-x} \leq -1$
- [ ] $0 \leq \dfrac{f(y)-f(x)}{y-x} \leq 2$

Question 5 : The integral $\displaystyle\int_0^1 \frac{2x-1}{(x-3)(x+2)}\,dx$ equals:

- [x] 0
- [ ] $\sqrt{6}\arctan\left(\frac{1}{6}\right)$
- [ ] -1
- [ ] $\log(3) - \log(2)$

Question 6 : Let $f\colon [\frac{1}{2},1] \to \mathbb{R}$ be the function defined by $f(x) = \frac{1}{x} + \frac{1}{\pi}\sin(\frac{\pi}{x})$. Let I be the range of f. Then:

- [ ] I=$[1, 1+\frac{1}{\pi}]$
- [ ] I=$[1-\frac{1}{\pi}, 1]$
- [ ] I=$[2,3]$
- [x] I=$[1,2]$

Question 7 : Let $\alpha \in \mathbb{R}$. The real series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{\alpha}{n}\right)^{n^2}$ converges if and only if

- [ ] $-1 < \alpha < 0$
- [ ] $\alpha < -1$
- [ ] $\alpha \geq 0$
- [x] $\alpha < 0$

Question 8 : The generalized integral $I = \displaystyle\int_{0^+}^{1}\frac{\log x}{x^2}dx$

- [ ] converges and equals -1
- [ ] converges and equals 1
- [ ] converges and equals -4
- [x] diverges

Question 9 :
The complex numbers $3, 1-2i$, and $1+2i$ are the roots (the zeros) of the polynomial

- [x] $z^3 - 5z^2 + 11z - 15$
- [ ] $z^3 - 2iz^2 + 45$
- [ ] $z^3 + 14z^2 + 15$
- [ ] $z^3 - 5z^2 + 5z + 45$

Question 10 : Let $a_0 \in \mathbb{R}$ and $(a_n)_{n\geq 0}$ a sequence of real numbers satisfying the following recurrence relation for $n \geq 1$

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{2}.$$

Then:

- [ ] if $a_0 < 0$, $\lim\limits_{n\to\infty} a_n = -\infty$
- [x] if $a_0 = 0$ the sequence is convergent
- [ ] if $a_0 > 1$ the sequence is increasing
- [ ] if $a_0 < 1$ the sequence is decreasing

Question 11 : Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n x^n$, where $a_n = 1$ if n is even and $a_n = 0$ if n is odd. The radius of convergence for the series, R, satisfies

- [ ] $R = \frac{1}{2}$
- [x] $R = 1$
- [ ] $R = \infty$
- [ ] $R = 0$

Question 12 : Let $(a_n)_{n\geq 1}$ be the sequence defined by

$$a_n = (3n+1)^{\log(\frac{1}{\sqrt{n}})}.$$

Then:

- ☐ $\lim_{n\to+\infty} a_n = 1$
- ☑ $\lim_{n\to+\infty} a_n = 0$
- ☐ $\lim_{n\to+\infty} a_n = +\infty$
- ☐ $\lim_{n\to+\infty} a_n = 3$

Question 13 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-2/|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

- ☐ $\lim_{x\to 0} f(x)$ exists but f is not continuous at 0
- ☐ $\lim_{x\to 0} f(x)$ does not exist
- ☐ f is continuous at $x=0$ but not differentiable at $x=0$
- ☑ f is differentiable at $x=0$

Question 14 : Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = e^x \log(1+x)$. The expansion of order 3 of f at $x_0 = 0$ is

- ☐ $f(x) = x + \frac{x^2}{2} + \frac{x^3}{2} + x^3\varepsilon_3(x)$
- ☐ $f(x) = x - \frac{x^2}{3} + \frac{x^3}{2} + x^3\varepsilon_3(x)$
- ☑ $f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + x^3\varepsilon_3(x)$
- ☐ $f(x) = x + \frac{x^2}{3} - \frac{x^3}{2} + x^3\varepsilon_3(x)$

Question 15 : Let the sequence $\{a_n\}_{n\geq 0}$ be defined by recurrence with $a_0 = \frac{3}{2}$ and for $n \geq 1$, $a_n = 3 - \frac{2}{a_{n-1}}$, then:

- ☐ the limit does not exist in $\mathbb{R}$
- ☐ $\lim_{n\to\infty} a_n = 1$
- ☑ $\lim_{n\to\infty} a_n = 2$
- ☐ $\lim_{n\to\infty} a_n = 4$

Question 16 : Let $a,\, b \in \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{\sqrt{2}}{2} & \text{si } x \leq 0\,, \\ \sin(ax+b) & \text{si } x > 0\,. \end{cases}$$

Then, f is continuous on $\mathbb{R}$ for :

☐ $a = \frac{\pi}{2}$ et $b = \frac{\pi}{2}$ ☐ $a = 0$ et $b = -\frac{\pi}{4}$ ☐ $a = -\frac{\pi}{4}$ et $b = 0$ ☑ $a = 0$ et $b = \frac{\pi}{4}$

Question 17 : For all $k \in \mathbb{N}$, $k \geq 1$, let $a_k = (-1)^k \dfrac{k+1}{k^2}$ and $s_n = \sum_{k=1}^{n} a_k$. Then

☑ the series $\sum_{k=1}^{\infty} a_k$ converges, but not absolutely.

☐ the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

☐ $\lim_{n\to\infty} s_n = \infty$.

☐ $\lim_{n\to\infty} s_n = -\infty$.

Question 18 : The integral $\int_0^1 x^2 e^{-x}\, dx$ equals

☑ $2 - \frac{5}{e}$ ☐ $2 - \frac{1}{e}$ ☐ $2 - \frac{3}{e}$ ☐ $2 - \frac{4}{e}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 19 : Let $(a_n)_{n\geq 0}$ be a sequence of non zero real numbers such that $\lim_{n\to\infty} a_n = 2$. Then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

☑ TRUE ☐ FALSE

Question 20 : Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijective increasing function. Then $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ is increasing.

☑ TRUE ☐ FALSE

Question 21 : The definite integral $\int_{-1}^{1} e^{-\sin(x)}\,dx$ is equal to zero.

☐ TRUE ☑ FALSE

Question 22 : Let $f \in C^\infty(\mathbb{R})$. Then for each $x_0 \in \mathbb{R}$ and each $n \in \mathbb{N}^*$ there is a Taylor expansion for f of order n around x_0.

☑ TRUE ☐ FALSE

Question 23 : Let $f \in C^1(\mathbb{R})$. Then there exist numbers $a, b \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{f(x) - a - bx}{x} = 0$$

☑ TRUE ☐ FALSE

Question 24 : For $z \in \mathbb{C}^*$, $z^5 + \frac{1}{z^5}$ is real if $|z| = 1$.

☑ TRUE ☐ FALSE

Question 25 : Let $f\colon [0,1] \to [0,1]$ be a continuous function with range $[0,1]$. Then, there must exist $x \in [0,1]$ such that $f(x) - x = 0$.

☑ TRUE ☐ FALSE

Question 26 : Let $A, B \subseteq \mathbb{R}$ be two non empty bounded subsets. If $\inf A \leq \inf B$ and $\sup A \geq \sup B$, then $B \subseteq A$.

☐ TRUE ☑ FALSE

Question 27 : Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous at $x_0 = 0$. Then, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = xf(x)$ is differentiable at $x_0 = 0$.

☑ TRUE ☐ FALSE

Question 28 : Let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ be two sequences of real numbers such that the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge. Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

☐ TRUE ☑ FALSE

Question 30: *This question is worth 5 points.*

□0 □1 □2 □3 □4 □5 *Do not write here.*

Show by induction that for each $n \geq 1$, $\sum_{k=1}^{n} 3^k(k+\frac{1}{2}) = 3^n \frac{3n}{2}$.

LET $H(n)$ BE $\sum_{k=1}^{n} 3^k(k+1/2) = 3^n \frac{3n}{2}$.

WE MUST SHOW a) $H(1)$ IS TRUE b) $H(n) \Longrightarrow H(n+1)$

a) (INITIAL STEP) IF $n=1$. LHS $= 3^1(1+1/2) = 3.\frac{3}{2}$

$= 3^1 \frac{3 \times 1}{2} =$ RHS.

b) (INDUCTIVE STEP) : SUPPOSE $H(n)$ IS TRUE.

THEN $\sum_{k=1}^{n+1} 3^k(k+1/2) = \sum_{h=1}^{n} 3^h(h+1/2) + 3^{n+1}(n+1+1/2)$

$= \sum_{n=1}^{n} 3^h(k+1/2) + 3^{n+1}(n+3/2)$

$\underline{H(n)}$ $3^n\left(\frac{3n}{2}\right) + 3^{n+1}(n+3/2) = 3^{n+1}\left(\frac{n}{2} + n+3/2\right)$

$= 3^{n+1}\left(\frac{3n}{2} + 3/2\right) = 3^{n+1} \frac{3(n+1)}{2}$.

ie $H(n+1)$ IS TRUE

Third part, open questions

Answer in the empty space below. Your answer should be carefully justified, and all the steps of your argument should be discussed in details. Leave the check-boxes empty, they are used for the grading.

Question 29: *This question is worth 6 points.*

☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 *Do not write here.*

(a) State the Bolzano Weierstrass theorem.

(b) Give a result for continuous functions on closed bounded intervals whose proof relies on the Bolzano Weierstrass Theorem. (No proof required.)

(c) Give an example of two sequences $\{a_n\}_{n>0}$, $\{b_n\}_{n>0}$ for which $\limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

a) IF $(a_n)_{n\geq 1}$ IS A BOUNDED SEQUENCE THEN $\exists$ A CONVERGENT SUBSEQUENCE OF $(a_n)_{n\geq 1}$

b) IF f IS A CTS FUNCTION ON A CLOSED BOUNDED. INTERVAL $\underline{I}$ THEN $\sup_{x\in I} f(x) < \infty$.

(OR $\exists$ $x_0 \in I$ SO THAT $f(x_0) = \sup_{x\in I} f(x)$

OR f IS UNIFORMLY CTS).

c). LET $a_n = (-1)^n$ $b_n = (-1)^{n+1}$ $n>0$.

THEN $a_n + b_n = 0$ $\forall n$ SO

$\limsup (a_n + b_n) = 0 < \limsup a_n + \limsup b_n$

$= 1 + 1$

Question 31: *This question is worth 5 points.*

0 1 2 3 4 5 *Do not write here.*

Show that the equation $cos(x) + x = 5$ has a unique solution in $\mathbb{R}$.

$f(x) = \cos(x) + x$ HAS $f'(x) = -\sin(x) + 1 \geq 0$.

SO $f'(x) = 0$ ONLY AT $\pi/2 + 2n\pi$. OTHERWISE IT IS STRICTLY POSITIVE.

SO BY MVT FOR DIFF FUNCTIONS.

f IS STRICTLY INCREASING ON INTERVALS

$I_n = [\pi/2 + 2n\pi, \pi/2 + 2(n+1)\pi]$

SO f IS STRICTLY INCREASING ON $\bigcup I_n = \mathbb{R}$.

IE f IS STRICTLY INCREASING SO $\exists$ AT MOST 1 VALUE IN $\mathbb{R}$ WITH $f(x) = 5$. $(*)$.

BUT $f(0) = 1$ $f(7) \geq 6$ SO BY THE INTERMEDIATE VALUE THEOREM FOR CTS FUNCTIONS $\exists$ $x \in]0,7[$ WITH $f(x) = 5$.

BY $(*)$ THIS x IS UNIQUE IN $\mathbb{R}$. $\square$